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On the estimation of the Weibull modulus

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It is well documented that nominally identical specimens of brittle materials, e.g. ceramics, show a large variation of tensile fracture stresses and in order to use brittle materials as engineering materials the strength has to be characterized. The most widely used expression for characterization is the cumulative distribution function proposed by Weibull [1]. The Weibull function is also known to statisticians as Fisher–Tipper Type III distribution of smallest values or as the third asymptotic distribution of smallest extreme value [2]. The Weibull statistics is based on the the "weakest link-hypothesis" which means that the most serious flaw in the specimen will control the strength. The most serious flaw is not necessarily the largest one because its severity also depends on where it is situated. In other words, the flaw which is subjected to the highest stress intensity factor will be strength controlling. The flaws initiating fracture can conveniently be classified as intrinsic or extrinsic [3]. The intrinsic flaws are introduced during fabrication and are predominantly inclusions and voids. The extrinsic flaws are stress-induced cracks, such as surface cracks introduced during machining and microcracks resulting from large residual stresses, e.g. due to thermal contraction anisotropy.

From an engineering point of view the strength variability of brittle materials requires a new design approach [4] and the following must be considered.

1. The aim of total safety of a component must be relaxed and a definite acceptable failure probability must be specified.

2. Due to the statistical variation of flaw-sizes the failure probability will increase with increasing component volume.

3. As discussed above, the failure of a complex component is not necessarily initiated at the point of the highest nominal stress.

In order to predict the reliability of a structure both the stress and strength distribution functions, which might be time dependent, have to be known [2]. The time dependence of strength can for instance at elevated temperatures be caused by creep crack growth, cavitation and pit formation [5]. Even time independent strength distributions can be complex due to multiple flaw populations in the same specimen [6].

In order to characterize the strength of a ceramic material, bend tests are usually performed and a distribution of strength values is obtained. For small sample sizes $\lesssim 50$ specimens the classical problem is how to relate the fraction of specimens failing below a given stress to Weibull's cumulative distribution function. Many different relations have been proposed to define the cumulative failure probability and the object of this paper is therefore to study the effects of some, more or less, common estimators upon the Weibull parameters for different sample sizes. A single flaw population and a time independent strength will be assumed.

The Weibull distribution is given by

$$P = 1 - \exp\left[-V\left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m\right] \qquad (1)$$

where P is the fracture probability for the stress σ, m is known as the Weibull modulus, V is the volume of the specimen, σ_0 is a scaling parameter and σ_u is a threshold stress below which the failure probability is zero. In this equation it is assumed that compressive stresses do not contribute to fracture. Furthermore, it is also assumed that the failure initiating flaws are volume distributed and that the tensile stress is constant in the voulme. The problem with a nonuniformly stressed body is, however, simply solved by replacing the "risk of rupture", $V[(\sigma - \sigma_u)/\sigma_0]^m$, with a volume integral. The expression for failure initiating surface flaws is obtained by replacing volume with area.

The mean strength of the distribution (i.e. Equation 1) is given by

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$$\bar{\sigma} = \sigma_u + \frac{\sigma_0 \Gamma(1 + 1/m)}{V^{1/m}} \quad (2)$$

where Γ is the gamma function.

The distribution can now be written as

$$P = 1 - \exp\left\{-\left[\Gamma\left(1 + \frac{1}{m}\right)\left(\frac{\sigma - \sigma_u}{\bar{\sigma} - \sigma_u}\right)\right]^m\right\} \quad (3)$$

It is often quoted that the mean strengths of two components, 1 and 2, with different volumes are related as

$$\left(\frac{\bar{\sigma}_1 - \sigma_u}{\bar{\sigma}_2 - \sigma_u}\right) = \left(\frac{V_2}{V_1}\right)^{1/m} \quad (4)$$

This relation is directly a consequence of Equation 2. However, as discussed by Batdorf [7] the weakest link theory does not actually permit us to draw this conclusion. The weakest link theory just tells that the risk of rupture will be directly related to the volume. However, it there is only one single flaw (strength) population in the specimen, then Equation 4 is valid.

The three-parameter function (i.e. Equation 1) can be used when the material can be guaranteed to have a minimum strength, for example obtained by proof testing in an inert atmosphere. But due to the danger in underestimating the safety factor it is recommended in general to put $\sigma_u = 0$ for brittle materials [8]. The following discussion, therefore, is confined to the two-parameter function. (Through a simple linear transformation the three-parameter function can be obtained from the two-parameter function [2] and the results later obtained do not lose anything in generality.)

By rearrangement and putting $\sigma_u = 0$ Equation 3 can be written as

$$y = \ln\left[\ln\left(\frac{1}{1-P}\right)\right] = m \ln \Gamma\left(1 + \frac{1}{m}\right) + m \ln\left(\frac{\sigma}{\bar{\sigma}}\right) \quad (5)$$

A plot of y aginst $\ln \sigma$ will evidently give a line of slope m. The slope can therefore be estimated graphically or for example by the least squares method or the maximum likelihood method. The results from any of these methods will give a biased estimate of m (e.g. [2 and 8]). The maximum likelihood method gives a m estimate, which is biased but is independent of the plotting procedure used. However, there are no obvious statistic arguments for choosing this method instead of the less expensive least squares method, at least for small sample sizes [8]. Therefore, the interest will be directed towards the latter method, in spite of the problems associated with the choice of an estimator for P. However, it must be realized that in using the least squares method it is assumed that the experimentally obtained $\ln \sigma$ values are Gaussian distributed around the "true" line given by Equation 5. It is probably more reasonable to assume that the experimentally obtained σ values are Gaussian distributed. This approach however, requires a direct nonlinear least squares curve fit and is a subject for future work.

Let us choose a sample size of n specimens and order the strength values so that

$$\sigma_1 \leqslant \sigma_2 \leqslant \ldots \sigma_j \ldots \ldots \leqslant \sigma_n$$

The following two estimators have been used by Trustrum and Jayatilaka [8] to calculate P_j for the jth strength

$$P_j = \frac{j}{n+1} \quad (6)$$

$$P_j = \frac{j - 0.5}{n} \quad (7)$$

The contender in Equation 6 is known as the mean rank value [2] and that in Equation 7 is the average value of the empirical density function before and after the jump at σ_j [8]. As demonstrated by [8] for sample sizes less than 50 specimens Equation 6 gives a more biased m estimate than Equation 7 and with approximately identical variances. The latter estimator is therefore to be preferred, a conclusion also drawn by Johnson [6].

Among other estimators discussed by [2] we have the median rank value which can be approximated by

$$P_j = \frac{j - 0.3}{n + 0.4} \quad (8)$$

A further example is

$$P_j = \frac{j - 3/8}{n + 1/4} \quad (9)$$

As cited by [2], White, who studied samples of size 6, found that Equation 9 gave the least-biased m estimate and that the most biased estimate resulted from Equation 6, while Equations 8 and 7 gave approximately equivalent m estimates.

These four contenders will now be compared for different sample sizes not greater than 50 specimens, because larger sample sizes are not often used in practice.

The statistical properties of the four estimators have been studied by a Monte Carlo simulation technique as described by Trustrum and Jayatilaka [8]. This procedure will now be shortly discussed. Let us start with rewriting Equation 5 as

$$\sigma = \left[\ln\left(\frac{1}{1-P}\right)\right]^{1/m} \tag{10}$$

where for sake of clarity $\bar{\sigma}$ has been put equal to $\Gamma(1 + 1/m)$. If we regard a large "specimen" population with a prescribed m value, it is seen that random strength values can be obtained from Equation 10 provided random number between 0 and 1 are substituted for the fracture probability, P. A computer program was written, which used a sample of random numbers to obtain strength values $\sigma_1, \sigma_2 \ldots \sigma_n$ for a given m value. The strength values were then ranked in ascending order. This was followed by a least squares analysis of the linear equation given in Equation 5 and the obtained specific m_i value of this sample was stored. The procedure was repeated for each of the four estimators between 2000 and 4000 times depending on the sample size. The generated random samples were of size $n = 10$, 20, 30, 40 and 50. The random strengths were generated from a distribution with $m = 10$. As has been shown by Bain and Antle [9] only one value of m needs to be considered for each estimator to obtain the bias and standard deviation of the "experimentally" determined m value. The mean value of m, $\bar{m}$, and the standard deviation of m, S_m, are obtained from

$$\bar{m} = \sum_{i=1}^{k} \frac{m_i}{k} \tag{11}$$

and

$$S_m^2 = \sum_{i=1}^{k} \frac{(m_i - \bar{m})^2}{k-1} \tag{12}$$

where k is the number of samples and m_i is the m value of sample i. In Table I the results of the Monte Carlo simulation for the different estimators are given. The listed values are $\bar{m}/m$ and $S_m/\bar{m}$ in parenthesis. For an unbiased m value $\bar{m}/m$ is expected to be close to unity.

By using the law of propagation of errors it has been shown that the coefficient of variation of m, i.e. the standard deviation of m divided by its mean value should be equal to $1/n^{1/2}$ [9]. In the last column of the table $1/n^{1/2}$ is given for the different sample sizes and as is seen this gives a good description of the statistically obtained $S_m/\bar{m}$ values. It should be noted that these figures as expected are larger than those given by the Cramer–Rao lower bound for unbiased estimates, which is $0.78/n^{1/2}$ [8].

From Table I it can be seen that the popular $P_j = j/(n + 1)$ gives the largest bias and that $\bar{m}/m$ increases with increasing sample size. It is also seen that the other estimators give a much smaller bias and that $\bar{m}/m$ in most cases show just a little deviation from the true value of one. The standard deviation of m for the different estimators is approximately equal and it is as expected decreasing with increasing sample size.

The results of the two first estimators are close to the results given by [8].

Though not shown in this paper it was found that the mean value of $\bar{\sigma}/\Gamma(1 + 1/m)$ in Equation 5 was close to one and that its coefficient of variation was less than 2%. It must be pointed out that the latter result was an effect of the chosen m value of 10. As it has already been shown [9], by using the law of propagation of errors, this coefficient of variation decreases with increasing sample size and m value.

One of the problems of performing an experiment in oreder to characterize the strength of a brittle material is to determine the number of specimens needed to get a good statistical reproducibility. The performed Monte Carlo simulation has shown that there is little if any difference in the coefficient of variation of m between the used

TABLE I Estimated means and standard deviations of m for four different estimators and different sample sizes. The last column shows the theoretical coefficient of variation of m

n	$P_j = j/(n+1)$		$P_j = (j-0.5)/n$		$P_j = (j-0.3)/(n+0.4)$		$P_j = (j-3.8)/(n+1/4)$		$1/n^{1/2}$
	$\bar{m}/m$	$(S_m/\bar{m})$	$\bar{m}/m$	$(S_m/\bar{m})$	$\bar{m}/m$	$(S_m/\bar{m})$	$\bar{m}/m$	$(S_m/\bar{m})$	
10	0.869	(0.333)	1.062	(0.330)	0.978	(0.328)	1.010	(0.332)	(0.316)
20	0.890	(0.240)	1.011	(0.230)	0.963	(0.226)	0.986	(0.228)	(0.224)
30	0.908	(0.189)	1.006	(0.186)	0.961	(0.185)	0.977	(0.187)	(0.183)
40	0.918	(0.167)	1.002	(0.166)	0.969	(0.164)	0.977	(0.162)	(0.158)
50	0.927	(0.149)	0.998	(0.143)	0.965	(0.148)	0.978	(0.144)	(0.141)

TABLE II Experimentally obtained and corrected m values for silicon nitride tested in three point bending

	$P_j = j/(n+1)$	$P_j = (j-0.5)/n$	$P_j = (j-0.3)/(n+0.4)$	$P_j = (j-3/8)/(n+1/4)$
m	11.01	12.61	11.87	12.12
corrected m	12.37	12.47	12.32	12.29

estimators and that it can be obtained from $1/n^{1/2}$. It is also shown that $P_j = (j-0.5)/n$ gives the least biased m estimate provided $n \geqslant 20$. For smaller populations the fourth estimator is to be preferred. However, as can be seen the bias is always much less than the coefficient of variation.

In order to see the effects of the different estimators some of the experimental results of Katayama and Hattari [10] are used. The studied material was sintered silicon nitride and the tabulated strength values of 20 specimens tested in three point bending have here been used for the analysis. The m values obtained for the different estimators are given in Table II. The experimentally obtained m value will most probably correspond to the statistically evaluated mean value given in Table I. A corrected value, which is expected to be close to the true value, is obtained by dividing the experimental value by the $\bar{m}/m$ value of Table I. As can be seen the expected m value is about 12.4. The standard deviation is expected to be $12.4/20^{1/2} \approx 2.8$ according to the last column of Table I. Though not shown in Table II the ratio $\bar{\sigma}/\Gamma(1+1/m)$ evaluated from the data as expected was not significantly different for the four estimators and this means that the predicted failure probability at low stresses is largest for $P_j = j/(n+1)$ and smallest for $P_j = (j-0.5)/n$. The first expression evidently gives a conservative failure probability and should therefore, from an engineering point of view, be the best choice in reliability predictions. However, from a materials science point of view, the second one is to be preferred.

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